A kinetic scheme for unsteady pressurised flows in closed water pipes

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Abstract

The aim of this paper is to present a kinetic numerical scheme for the computations of transient pressurised flows in closed water pipes. Firstly, we detail the mathematical model written as a conservative hyperbolic partial differentiel system of equations, and then we recall how to obtain the corresponding kinetic formulation. Then we build the kinetic scheme ensuring an upwinding of the source term due to the topography performed in a close manner described by Perthame et al. [13, 1] using an energetic balance at microscopic level. The validation is lastly performed in the case of a water hammer in an uniform pipe: we compare the numerical results provided by an industrial code used at EDF-CIH (France), which solves the Allievi equation (the commonly used equation for pressurised flows in pipes) by the method of characteristics, with those of the kinetic scheme. It appears that they are in a very good agreement.

1 Introduction

The work presented in this article is the first step in a more general project: the modelisation of unsteady mixed water flows in open channels and in pipes, its kinetic formulation and its numerical resolution by a kinetic scheme.

Since we are interested in flows occuring in closed pipes, it may happen that some parts of the flow are free-surface (this means that only a part of the cross-section of the pipe is filled) and other parts are pressurised (this means that all the cross-section of the pipe is filled). Let us thus recall the current and previous works about mixed flows in closed water pipes by a partial state of the art review. Cunge and Wegner [8] studied the pressurised flow in a pipe as if it were a free-surface flow by assuming a narrow slot to exist in the upper part of the tunnel, the width of the slot being calculated to provide the correct sonic speed. This approach has been credited to Preissmann. Later, Cunge [7] conducted a study of translation waves in a power canal containing a series of transitions, including a siphon. Pseudoviscosity methods were employed to describe the movement of bores in open-channel reaches. Wiggert [16] studied the transient flow phenomena and his analytical considerations included open-channel surge equations that were solved by the method of characteristics. He subjected it to subcritical flow conditions. His solution resulted from applying a similarity between the movement of a hydraulic bore and an interface (that is, a surge front wave). Following Wiggert's model, Song, Cardle and Leung [14] developed two mathematical models of unsteady free-surface/pressurised flows using the method of characteristics (specified time and space) to compute flow conditions in two flow zones. They showed that the pressurised phenomenon is a dynamic shock requiring a full dynamic treatment even if inflows and other boundary conditions change very slowly. However the Song models

do not include the bore presence in the free-surface zone. Hamam and McCorquodale [11] proposed a rigid water column approach to model the mixed flow pressure transients. This model assumes a hypothetical stationary bubble across compression and expansion processes. Li and McCorquodale [12] extended the rigid water column approach to allow for the transport of the trapped air bubble. Recently Fuamba [9] proposed a model for the transition from a free surface flow to a pressurised. He wrote the conservation of mass, momentum and energy through the transition point and proposed a laboratory validation of his model. In the last few years, numerical models mainly based on the Preissmann slot technique have been developed to handle the flow transition in sewer systems. Implementing the Preissmann slot technique has the advantage of using only one flow type (free-surface flow) throughout the whole pipe and of being able to easily quantify the pressure head when pipes pressurise. Let us specially mention the work of Garcia-Navarro, Alcrudo and Priestley [10] in which an implicit method based on the characteristics has been proposed.

The Saint Venant equations, which are written in a conservative form, are usually used to describe free surface flows of water in open channels. As said before, they are also used in the context of mixed flows (i.e. either free surface or pressurized) using the artifice of the Preissmann slot [15],[6]. On the other hand, the commonly used model to describe pressurized flows in pipes is the system of the Allievi equations [15]. This system of 1st order partial differential equations cannot be written under a conservative form since this model is derived by neglecting some acceleration terms. This non conservative formulation is not appropriate for a kinetic interpretation of the transition between the two types of flows since we are not able to write conservations of appropriate quantities such as momentum and energy.

Then, it appears that a conservative model and a kinetic interpretation of it which describes pressurised flows in closed water pipes could be of great interest.

The model used in this article to describe pressurised flows in closed water pipe is very closed to the Shallow Water equations, and has been established by the authors in [4]. A second order well-balanced finite volume scheme was therein presented. We will recall in section 2 the main features of this previous work.

Another approach for the numerical resolution of Shallow Water equations is to use a kinetic formulation [13, 1]. The corresponding scheme appears to have interesting theoretical properties: the scheme preserves the still water steady state and involves a conservative in-cell entropy inequality. Moreover, this type of numerical approximation leads to an easy implementation. The present modelisation of pressurised flows is formally very close to the Shallow Water equations and it may be very interesting to propose a kinetic formulation and thus to construct a kinetic scheme.

The model for the unsteady mixed water flows in closed water pipes and a finite volume discretisation has been previoulsy studied by the authors [3] and a kinetic formulation has been proposed in [5]. We will recall in section 2 the main results and the properties of this kinetic formulation that will be useful to show the properties of the numerical kinetic scheme such as the preservation of the steady state water at rest, and the positivity of the wetted area.

Section 3 is devoted to the construction of the kinetic scheme. The upwinding of the source term due to the topography is performed in a close manner described by Perthame et al. [13] using an energetic balance at microscopic level for the Shallow Water equations.

Finally, we present in section 4 a numerical validation of this study by the comparison between the resolution of this model and the resolution of the Allievi equation solved by the research code `belier` used at Center in Hydraulics Engineering of Electricité De France (EDF) [17] for the case of critical waterhammer tests.

2 The mathematical model and the kinetic formulation

We derived a conservative model for pressurised flows from the 3D system of compressible Euler equations by integration over sections orthogonal to the flow axis.

2.1 The mathematical model : a "Shallow Water like" system of equations

The equation for conservation of mass and the first equation for the conservation of momentum are:

$$\partial_t \rho + \text{div}(\rho \vec{U}) = 0 \tag{1}$$

$$\partial_t(\rho \, u) + \text{div}(\rho \, u \vec{U}) = F_x - \partial_x P \tag{2}$$

with the speed vector $\vec{U} = u\vec{i} + v\vec{j} + w\vec{k} = u\vec{i} + \vec{V}$, where the unit vector $\vec{i}$ is along the main axis, ρ is the density of the water. We use the Boussinesq linearised pressure law (see [15]):

$$P = P_a + \frac{1}{\beta}\left(\frac{\rho}{\rho_0} - 1\right),$$

where ρ_0 is the density at the atmospheric pressure P_a and β the coefficient of compressibility of the water. Exterior strengths $\vec{F}$ are the gravity $\vec{g}$ and the friction term S_f which is assumed to be given by the Manning-Strickler law (see [15]):

$$S_f = K \, u \, |\, u \,| \quad \text{with} \quad K = \frac{1}{K_s^2 \, R_h^{4/3}} \tag{3}$$

where $K_s > 0$ is the Strickler coefficient, depending on the material, and R_h is the so called hydraulic radius given by $R_h = \dfrac{S}{P_m}$. S represents the cross-section area of the pipe whereas P_m is the perimeter of the section. Then Equations (1)-(2) become:

$$\partial_t \rho + \partial_x(\rho \, u) + \text{div}_{(y,z)}(\rho \vec{V}) = 0$$

$$\partial_t(\rho \, u) + \partial_x(\rho \, u^2) + \text{div}_{(y,z)}(\rho \, u \vec{V}) = -\rho g(\partial_x Z + S_f) - \frac{\partial_x \rho}{\beta \rho_0} \quad .$$

Assuming that the pipe is infinitely rigid and has a uniform constant cross-section S, and taking averaged values in sections orthogonal to the main flow axis, we get the following system written in a conservative form for the unknowns $M = \rho \, S$, $D = \rho \, S \, u$:

$$\partial_t(M) + \partial_x(D) = 0 \tag{4}$$

$$\partial_t(D) + \partial_x\left(\frac{D^2}{M} + c^2 \, M\right) = -g \, M(\partial_x Z + S_f) \tag{5}$$

where $c = \dfrac{1}{\sqrt{\beta \, \rho_0}}$ is the speed of sound. A complete derivation of this model, taking into account the deformations of the pipe, contracting or expanding sections, and a spatial second order Roe-like finite volume method in a linearly implicit version is presented in [4] (see [2] for the first order implicit scheme). This system of partial differential equation is formally close to the Shallow Water equations where the conservative variables are the wet area and the discharge, thus we define an "FS-equivalent" wet area (FS for Free Surface) A and a "FS-equivalent discharge" Q through the relations:

$$M = \rho \, S = \rho_0 \, A \quad \text{and} \quad D = \rho \, S \, u = \rho_0 \, Q \quad .$$

Dividing (4)-(5) by ρ_0 we can write this system under the conservative form:

$$\partial_t U + \partial_x F(U) = G(x, U) \tag{6}$$

where the unknown state is $U = (A, Q)^t$, the flux vector is $F(U) = (Q, \dfrac{Q^2}{A} + c^2 A)^t$ and the source term writes $G(x, U) = (0, -gA(\partial_x Z + S_f))^t$. This new set of variables allows a more natural treatment of mixed flows (see [4]).

Let us now recall the main properties of the system (6) whose proofs can be found in [5].

Theorem 1 *The system (6) is strictly hyperbolic. It admits a mathematical entropy:*

$$E(A, Q, Z) = \frac{Q^2}{2A} + gAZ + c^2 A \ln A \tag{7}$$

which satisfies, for smooth enough solution, the entropy inequality :

$$\partial_t E + \partial_x[u(E + c^2 A)] \leq 0 \quad .$$

Also, for the frictionless pipes ($S_f = 0$), the system (6) admits a family of smooth steady states characterized by the relations:

$$Q = Au = C_1 \ ,$$

$$\frac{u^2}{2} + g\,Z + c^2 \ln A = C_2 \ ,$$

where C_1 and C_2 are two arbitrary constants. The quantity $\dfrac{u^2}{2} + g\,Z + c^2 \ln A$ is also called the total head.

Remark 1 An easy computation leads to the equality:

$$\partial_t E + \partial_x[u(E + c^2 A)] = -uS_f = -K|u|^3 \leq 0 \ .$$

Thus for a frictionless pipe we obtain an entropy equality whereas the entropy inequality is strict as soon as a friction term is considered.

Let us also remark that the still water steady state for frictionless pipe, namely $u \equiv 0$, satisfies: $g\,Z + c^2 \ln A = C_2$.

2.2 The kinetic approach

We present in this section the kinetic formulation for pressurised flows in closed water pipes modelised by the preceding system of partial differential equations (see [5] for more details and properties). Let us mention that the following results (namely Theroem 2 and Theorem 3) are only valid for frictionless pipes. Let us consider a smooth real function χ which has the following properties:

$$\chi(\omega) = \chi(-\omega) \geq 0 \ , \ \int_{\mathbb{R}} \chi(\omega)d\omega = 1, \ \int_{\mathbb{R}} \omega^2 \chi(\omega)d\omega = 1 \ . \tag{8}$$

We then define the density of particles $\mathcal{M}(t, x, \xi)$ by the so-called *Gibbs equilibrium*:

$$\mathcal{M}(t, x, \xi) = \frac{A(t, x)}{c} \chi\left(\frac{\xi - u(t, x)}{c}\right) \quad .$$

These definitions allow to obtain a kinetic representation of the system (6) by the following result (see [5] for the proof).

Theorem 2 *The couple of functions (A, Q) is a strong solution of the system (6) if and only if $\mathcal{M}$ satisfies the kinetic equation:*

$$\frac{\partial}{\partial t}\mathcal{M} + \xi \cdot \frac{\partial}{\partial x}\mathcal{M} - g\frac{\partial}{\partial x}Z \cdot \frac{\partial}{\partial \xi}\mathcal{M} = K(t, x, \xi) \tag{9}$$

for some collision term $K(t, x, \xi)$ which satisfies for a.e. (t, x)

$$\int_{\mathbb{R}} K \, d\xi = 0 \,, \quad \int_{\mathbb{R}} \xi \, K d\xi = 0 \quad.$$

This result is a consequence of the following relations verified by the microscopic equilibrium:

$$A = \int_{\mathbb{R}} \mathcal{M}(\xi) \, d\xi \,, \tag{10}$$

$$Q = \int_{\mathbb{R}} \xi \mathcal{M}(\xi) \, d\xi \,, \tag{11}$$

$$\frac{Q^2}{A} + c^2 A = \int_{\mathbb{R}} \xi^2 \mathcal{M}(\xi) \, d\xi \,. \tag{12}$$

This theorem produces a very useful consequence: the nonlinear system (6) can be viewed as a simple linear equation on a nonlinear quantity $\mathcal{M}$ for which it is easier to find simple numerical schemes with good theoretical properties: it is this feature which will be exploited to construct a kinetic scheme.

Theorem 3 *Let $A(x, t) > 0$ and $Q(x, t)$ be two given functions.*

1. The minimum of the energy:

$$\mathcal{E}(f) = \int_{\mathbb{R}} \left(\frac{\xi^2}{2} f(\xi) + c^2 f(\xi) \ln(f(\xi)) + gZf(\xi) + c^2 \ln(c\sqrt{2\pi}) f(\xi) \right) d\xi \quad,$$

under the constraints:

$$f \geq 0 \,, \quad \int_{\mathbb{R}} f(\xi) d\xi = A \,, \quad \int_{\mathbb{R}} \xi f(\xi) d\xi = Q \quad,$$

is attained by the function:

$$\mathcal{M}(t, x, \xi) = \frac{A}{c} \chi \left(\frac{\xi - u(t, x)}{c} \right)$$

where χ is defined by:

$$\chi(\omega) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\omega^2}{2}\right) \quad. \tag{13}$$

2. Moreover, the function χ defined by (13) ensures us to have the relation

$$\mathcal{E}(\mathcal{M}) = E(A, Q, Z)$$

if A and Q are solution of the pressurised flow equations (6) and the entropy E is defined by (7).

3. The Gibbs equilibrium $\mathcal{M}$ satisfies the still water steady state equation.

3   The kinetic scheme

The spatial domain is a pipe of length L. The main axis of the pipe is divided in N meshes $m_i =]x_{i-1/2}, x_{i+1/2}[$, $1 \leq i \leq N$, of length h_i and center x_i. We denote $\Delta x = \min_{1 \leq i \leq N} h_i$. Δt denotes the time step at time t_n and we set $t_{n+1} = t_n + \Delta t$.

The discrete macroscopic unknowns are $U_i^n = \begin{pmatrix} A_i^n \\ Q_i^n \end{pmatrix}$ with $1 \leq i \leq N$ and $0 \leq n \leq n_{max}$. They represent the mean value of U on the cell m_i at time t_n.

If $Z(x)$ is the function describing the bottom elevation, its piecewise constant representation is given by $\bar{Z}(x) = Z_i \mathbb{I}_{m_i}(x)$ with $Z_i = Z(x_i)$ for example.

Replacing Z by $\bar{Z}$ and neglecting the collision term $K(t, x, \xi)$ in a first step, the Equation (9) in the cell m_i writes:

$$\frac{\partial}{\partial t}\mathcal{M} + \xi \cdot \frac{\partial}{\partial x}\mathcal{M} = 0 \quad \text{for } x \in m_i \quad . \tag{14}$$

This equation is a linear transport equation whose explicit discretisation may be done directly by the following way. Denoting for $x \in m_i$, $f(t_n, x, \xi) = \mathcal{M}_i^n(\xi)$ the maxwellian state associated to A_i^n, and Q_i^n, the usual finite volume discretisation of the Equation (14) leads to:

$$f_i^{n+1}(\xi) = \mathcal{M}_i^n(\xi) + \frac{\Delta t}{h_i}\xi \left(\mathcal{M}_{i+\frac{1}{2}}^-(\xi) - \mathcal{M}_{i-\frac{1}{2}}^+(\xi) \right) \tag{15}$$

where the fluxes $\mathcal{M}_{i+\frac{1}{2}}^\pm$ have to take into account the discontinuity of the altitude $\bar{Z}$ at the cell interface $x_{i+1/2}$. Indeed, noticing that the fluxes can also be written as:

$$\mathcal{M}_{i+\frac{1}{2}}^-(\xi) = \mathcal{M}_{i+\frac{1}{2}} + \left(\mathcal{M}_{i+\frac{1}{2}}^- - \mathcal{M}_{i+\frac{1}{2}} \right)$$

the quantity $\delta\mathcal{M}_{i+\frac{1}{2}}^- = \mathcal{M}_{i+\frac{1}{2}}^- - \mathcal{M}_{i+\frac{1}{2}}$ holds for the discrete contribution of the source term $gA\partial_x Z$ in the system for negative velocities $\xi \leq 0$ due to the upwinding of the source term. Thus $\delta\mathcal{M}_{i+\frac{1}{2}}^-$ has to vanish for positive velocity $\xi > 0$, as proposed by the choice of the interface fluxes below.

Let us now detail our choice for the fluxes $\mathcal{M}_{i+\frac{1}{2}}^\pm$ at the interface. It can be justified by using a generalised characteristic method for the Equation (9) (without the collision kernel) but we give instead a presentation based on some physical energetic balance. Let us denote $\Delta^- Z_{i+\frac{1}{2}} = Z_{i+1} - Z_i$ and $\Delta^+ Z_{i+\frac{1}{2}} = Z_i - Z_{i+1}$. In order to take into account the neighboring cells by means of a natural interpretation of the microscopic features of the system, we formulate a peculiar discretisation for the fluxes in (15), computed by the following upwinded formulas:

$$\mathcal{M}_{i+\frac{1}{2}}^-(\xi) = \mathcal{M}_i^n(\xi)\,\mathbb{I}_{\xi \geq 0} + \overbrace{\mathcal{M}_i^n(-\xi)\,\mathbb{I}_{\xi^2 \leq 2g\Delta^- Z_{i+\frac{1}{2}}}\,\mathbb{I}_{\xi \leq 0}}^{reflection} \tag{16}$$
$$+ \underbrace{\mathcal{M}_{i+1}^n\left(-\sqrt{\xi^2 - 2g\Delta^- Z_{i+\frac{1}{2}}}\right)\,\mathbb{I}_{\xi^2 \geq 2g\Delta^- Z_{i+\frac{1}{2}}}\,\mathbb{I}_{\xi \leq 0}}_{transmission}$$

$$\mathcal{M}_{i+\frac{1}{2}}^+(\xi) = \mathcal{M}_{i+1}^n(\xi)\,\mathbb{I}_{\xi \leq 0} + \overbrace{\mathcal{M}_{i+1}^n(-\xi)\,\mathbb{I}_{\xi^2 \leq 2g\Delta^+ Z_{i+\frac{1}{2}}}\,\mathbb{I}_{\xi \geq 0}}^{reflection} \tag{17}$$
$$+ \underbrace{\mathcal{M}_i^n\left(\sqrt{\xi^2 - 2g\Delta^+ Z_{i+\frac{1}{2}}}\right)\,\mathbb{I}_{\xi^2 \geq 2g\Delta^+ Z_{i+\frac{1}{2}}}\,\mathbb{I}_{\xi \geq 0}}_{transmission}$$

The effect of the source term is made explicit by treating it as a physical potential. The choices (16)-(17) are thus a mathematical formalization to describe the physical microscopic behaviour of the system. The contribution of the interface $x_{i+1/2}$ to f_i^{n+1} is given by:

- the particles in the cell m_i at time t_n with non negative velocities ξ through the term $\mathcal{M}_i^n(\xi)\, \mathbb{I}_{\xi \geq 0}$ and those of them that are reflected (thus taken into account with velocity $-\xi$) if their kinetic energy is not large enough to overpass the potential difference i.e. $\xi^2 \leq 2g\Delta^{\pm}Z_{i+\frac{1}{2}}$: see Figure 2.

- the particles in the cell m_{i+1} at time t_n with a kinetic energy enough to overpass the potential difference ($\xi^2 \geq 2g\Delta^{\pm}Z_{i+\frac{1}{2}}$) and speed up or down according to this potential jump. It is the transmission phenomenon in classical mechanics as shown in Figure 1.
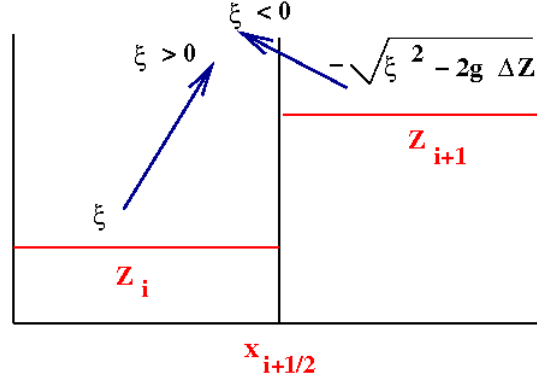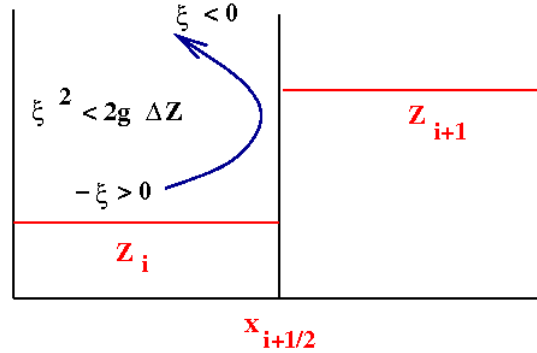

Figure 1: Transmission

Figure 2: Reflection

Since we neglected the collision term, it is clear that f^{n+1} computed by the discretised kinetic equation (15) is no more a Gibbs equilibrium. Therefore, to recover the macroscopic variables A and Q, according to the identities (10)-(11), we set:

$$U_i^{n+1} = \begin{pmatrix} A_i^{n+1} \\ Q_i^{n+1} \end{pmatrix} \stackrel{def}{=} \int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} f_i^{n+1}\, d\xi \tag{18}$$

Now, we can integrate the discretised kinetic equation (15) against 1 and ξ to obtain the macroscopic kinetic scheme:

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{h_i}\left(F_{i+\frac{1}{2}}^- - F_{i-\frac{1}{2}}^+ \right) \tag{19}$$

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The numerical fluxes are thus defined by the kinetic fluxes as follows:

$$F^{\pm}_{i+\frac{1}{2}} \overset{def}{=} \int_{\mathbb{R}} \xi \begin{pmatrix} 1 \\ \xi \end{pmatrix} \mathcal{M}^{\pm}_{i+\frac{1}{2}}(\xi)\, d\xi \qquad (20)$$

Remark 2

- We see immediately that the kinetic scheme (19)-(20) is wetted area conservative. Indeed, let us denote the first component of the discrete fluxes (20) $(F_A)^{\pm}_{i+\frac{1}{2}}$:

$$(F_A)^{\pm}_{i+\frac{1}{2}} \overset{def}{=} \int_{\mathbb{R}} \xi \mathcal{M}^{\pm}_{i+\frac{1}{2}}(\xi)\, d\xi$$

  An easy computation using the change of variables $\mu = \mid \xi \mid^2 - 2g\Delta^+ Z_{i+\frac{1}{2}}$ in the formulas (16)-(17) defining the kinetic fluxes $\mathcal{M}^{\pm}_{i+\frac{1}{2}}$ allows us to show that:

$$(F_A)^{+}_{i+\frac{1}{2}} = (F_A)^{-}_{i+\frac{1}{2}}$$

- Computing the macroscopic state U by the formula (18) or the fluxes by the formula (20) is not easy if the function χ verifying the properties (8) is not compactly supported. We use instead the function defined by:

$$\chi(\omega) = \frac{1}{2\sqrt{3}} \, \mathbb{1}_{[-\sqrt{3},\sqrt{3}]}(\omega)\ . \qquad (21)$$

  We get $\mathcal{M}^{n}_{i}(\xi) = \dfrac{A^{n}_{i}}{2\,c\,\sqrt{3}} \, \mathbb{1}_{[u^{n}_{i} - c\sqrt{3},\, u^{n}_{i} + c\sqrt{3}]}(\xi)$. Of course, from Theorem 3, the property on the microscopic energy and the still water steady state is no more valid but we will prove in Theorem 4 that our proposed kinetic scheme preserves the still water steady state (for the frictionless pipes) and the positivity of the equivalent wetted area.

- In the case where the friction S_f defined by (3) is present, it is added at the macroscopic level (19) as an extra source term.

We are now able to state the main properties of the kinetic scheme.

Theorem 4 *We choose the function χ defined by the formula (21) and we assume the CFL condition*

$$\Delta t \max_{1 \leq i \leq N} \left(\mid u^{n}_{i} \mid + c\sqrt{3} \right) \leq \Delta x. \qquad (22)$$

Then

  (i) the kinetic scheme (19)-(20) keeps the pseudo wetted area A^{n}_{i} positive.

  (ii) the kinetic scheme (19)-(20) preserves the still water steady state,

$$u^{n}_{i} = 0\,,\ g\,Z_i + c^2 \ln A_i = K$$

Proof of theorem 4 Let us mention that the CFL condition (22) is obtained from the linear discretised kinetic transport equation (15) for the particular choice of the function χ defined by the formula (21). This condition ensures the positivity of the pseudo wetted area as we show below.

Since $A_i = \int_{\mathbb{R}} f_i^{n+1} \, d\xi$, it is sufficient to prove that $f_i^{n+1} \geq 0$. Writing the microscopic scheme (15), (16), (17), using the CFL condition (22), and the fact that the function χ that we have chosen is compactly supported, one may see that if we suppose that $A_i^n \geq 0$, then f_i^{n+1} is a sum of non-negative quantities. For the second point, setting $u_i^n = 0$, we prove easily that in the discretised kinetic equation (15), we have

$$\mathcal{M}_{i+\frac{1}{2}}^{-}(\xi) = \mathcal{M}_{i-\frac{1}{2}}^{+}(\xi).$$

This implies $f_i^{n+1} = \mathcal{M}_i^n(\xi)$, which ensures by definition $A_i^{n+1} = A_i^n$ and $Q_i^{n+1} = Q_i^n$. Thus we obtain $u_i^{n+1} = 0$.

4 Numerical validation

We present now numerical results of a water hammer test. The pipe of circular cross-section of 2 m^2 (the diameter therefater is denoted by δ) and thickness $e = 20$ cm is 2000 m long. The altitude of the upstream end of the pipe is 250 m and the slope is 5°. The Young modulus is $23 \, 10^9$ Pa since the pipe is supposed to be built in concrete.
The density at the atmospheric pressure ρ_0 is 1000 kg/m^3 and the coefficient of compressibility of the water β is $5.0 \, 10^{-10}$ Pa^{-1}.
The wave speed is thus obtained by the formula (see [15, formula (2.39)]):

$$a = \frac{c}{\sqrt{1 + \dfrac{\delta}{\beta \, e \, E}}} = 1086.6 \text{ m/s}^{-1} \quad . \tag{23}$$

The total upstream head is 300 m. The initial downstream discharge is 10 m^3/s and we cut the flow in 5 seconds. Let us define the piezometric line by:

$$piezo = z + \delta + p \quad \text{with } p = \frac{c^2 \, (\rho - \rho_0)}{\rho_0 \, g} \quad . \tag{24}$$

We present a validation of the proposed scheme by comparing numerical results of the proposed model solved by the kinetic scheme with the ones obtained by solving Allievi equations by the method of characteristics with the so-called `belier` code: an industrial code used by the engineers of the Center in Hydraulics Engineering of Electricité De France (EDF) [17]. Our code is written in Fortran and runs during a few seconds on LinuX, Windows and MacIntosh operating systems.

A simulation of the water hammer test was done for a CFL coefficient equal to 0.8 (i.e. $CFL = 0.8$) and a spatial discretisation of 1000 mesh points (the mesh size is equal to 2 m). In the Figure 3, we present a comparison between the results obtained by our kinetic scheme and the ones obtained by the "belier" code at the middle of the pipe: the behavior of the piezometric line, defined by Equation (24), and the discharge at the middle of the pipe. One can observe that the results for the proposed model are in very good agreement with the solution of Allievi equations. In Figure 4, we present the piezometric line for the beginning and the end of the simulation. One can see that the peak of pressure (observed in the beginning of the simulation) is very well obtained by the kinetic scheme. Thus the strength of the water hammer is very good predicted by the proposed numerical scheme. A little smoothing effect, observed at the end of the simulation, may be probably due to the first order discretisation type. A second order scheme could be implemented naturally and produce a better approximation.

5 Conclusion

As mentionned in the introduction, our goal is to build a kinetic scheme for mixed flows. Perthame et al. [1, 13] have shown that the kinetic approach is relevant for free surface flows in open channels: the resulting kinetic scheme is easily implemented an enjoyed very good properties (posivity of the wetted area and discrete entropy inequalities). For pressurised flows, we have shown that this approach is also relevant. This allows us to investigate the construction of a kinetic scheme for mixed flows in closed water pipes.

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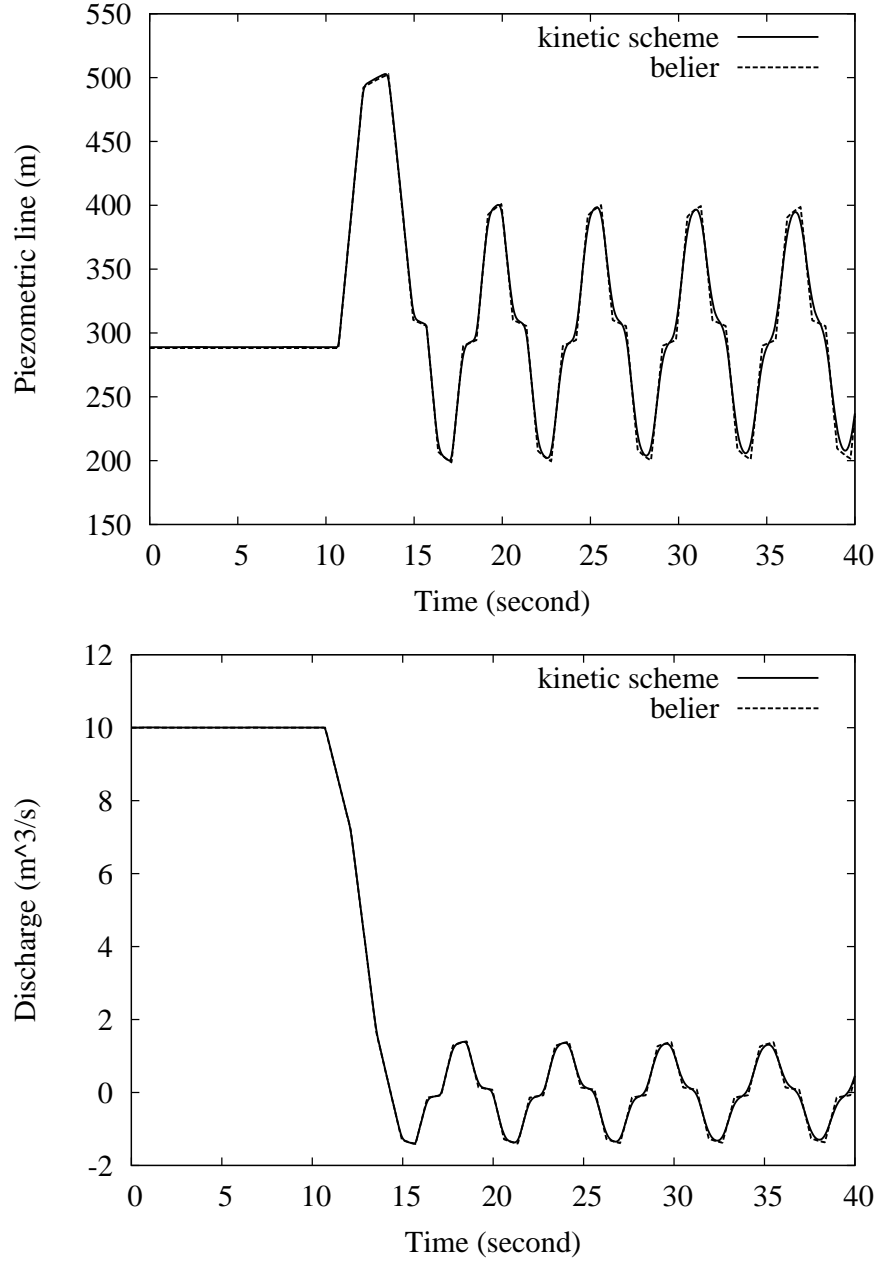
Figure 3: Comparison between the kinetic scheme and the industrial code belier
Piezometric line (top) and discharge (bottom) at the middle of the pipe
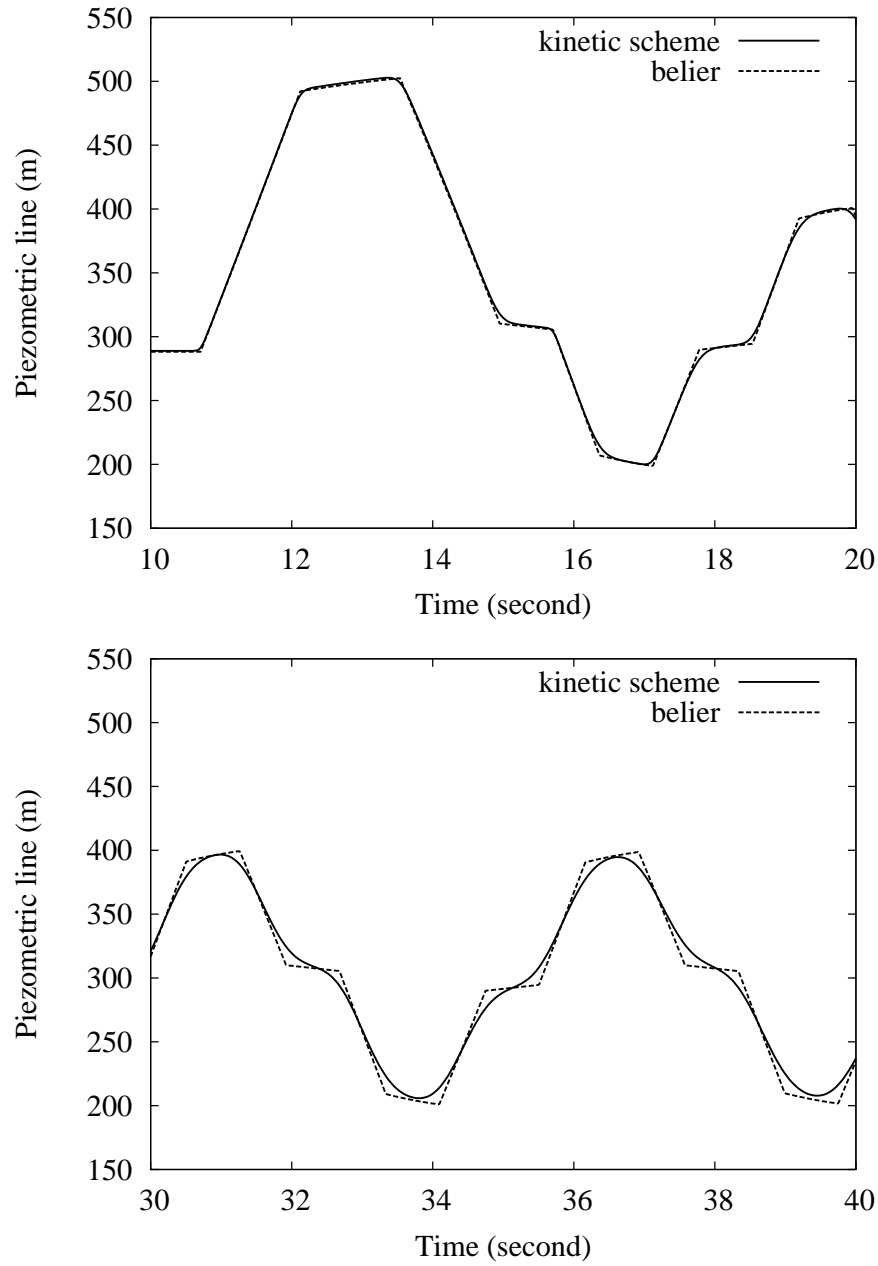
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Figure 4: Comparison between the kinetic scheme and the industrial code belier
Beginning of the simulation (top) and end of the simulation (bottom) at the middle of the pipe

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